CHAOS WITHOUT NONLINEAR DYNAMICS

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ABSTRACT

Recently, it has been shown that chaos can be synthesized by the linear superposition of certain pulse basis functions. Here, we extend this result and show that a linear, second-order filter driven by a random signal can generate a waveform that is chaotic under time reversal. That is, the waveform exhibits determinism and a positive Lyapunov exponent when viewed backward in time. We demonstrate the filter using a passive electronic circuit, and the resulting waveform exhibits a Lorenz-like butterfly structure. This method for generating chaotic waveforms may be useful for a number of potential applications, including spread-spectrum communication and ultra-wideband (UWB) radar and ladar. The filter also demonstrates that chaos may be connected to physical theories beyond those described by a deterministic nonlinear dynamical system.

1. INTRODUCTION

Chaotic waveforms have been suggested for a number of applications, including spread-spectrum communication and ultra-wideband (UWB) radar and ladar. For these applications, one usually assumes a nonlinear dynamical system is required to generate the chaotic waveform. However, it has recently been shown that chaos can also be constructed by linear superposition of special basis pulses [Hayes, 2003, 2005; Hirata and Judd, 2005; Corron *et al.*, 2006]. This surprising result implies that chaos can also be formally generated using a linear filter. However, such linear synthesis of chaos does not appear to be practical for a physical system. The pulse basis functions contain an infinitely long, exponentially increasing oscillation that culminates in a large central pulse and monotonic exponential decay. Thus, the exact filter is necessarily acausal and impractical to realize.

In this paper, we exploit the fact that the basis pulses do not need to be acausal when viewed in reverse time and develop a very simple system for generating chaotic waveforms. In particular, we find that a linear, second-order filter driven by a random bipolar signal can generate a waveform that is chaotic in reverse time. We call such dynamics *reverse-time chaos*. That is, when viewed backward in time, the waveform exhibits the essential qualities of a chaotic waveform, including determinism and a positive Lyapunov exponent. We implement the filter in an electronic circuit using only a few passive linear components, and we obtain a waveform that exhibits a Lorenz-like butterfly structure [Lorenz, 1963].

From previous work, it is known that a linear filter can increase the apparent dimensionality of a chaotic signal [Badii *et al.*, 1988]. But here we show that a linear filter driven by a random process produces a waveform that appears to have been produced by a low-dimensional chaotic system. This phenomenon suggests that chaos may be connected to physical theories whose underlying framework is not that of a traditional deterministic nonlinear dynamical system. Furthermore, the simplicity of the filter mechanism implies one must allow for the possibility of reverse-time chaos occurring naturally. For example, efforts to detect determinism from randomness in a physical system may require the arrow of time to differentiate chaos from linear filtering of noise. Powerful algorithms that rely only on geometric analysis of state space structures, such as fractal dimension, template analysis [Gilmore, 1998], or false nearest neighbors [Kennel *et al.*, 1992], may not detect a difference between forward and reverse-time chaos.

Several potential technology applications exist for reverse-time chaos. For communications, symbolic dynamics can be encoded directly by modulating the polarity of the basis pulses [Hayes *et al.*, 1993]. At the receiver, the determinism of reverse-time chaos provides a form of intentional inter-symbol interference that can be processed using simple predictive filters [Blahut, 1990; Hayes, 2005]. For correlation-based UWB ranging using chaotic waveforms, we note that reverse-time waveforms will work equally well as forward-time chaos [Wu and Jaggard, 1999; Myneni *et al.*, 2001; Lin and Liu, 2004a, 2004b]. Reverse-time chaos may also be relevant to the prediction and control of electromagnetic interference in circuits driven by radio frequency signals [Carroll, 2003; de Moraes and Anlage, 2004].

2. REVERSE-TIME CHAOS

We first demonstrate the construction of reverse-time chaos using a simple, discrete-time linear filter. This overly simple model is still sufficient to illustrate the fundamental mechanism underlying the physically realizable filter described in the next section.

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We begin by considering the shift map

$$z_{n+1} = 2z_n \mod 1 \tag{1}$$

for an arbitrary initial condition $0 \le z_0 < 1$. It is well known that this simple dynamical system is chaotic [Hayes, 2005]. If the state z_n is written as a binary fraction, the map corresponds to a left shift followed by dropping the integer bit. For example, the initial condition $z_0 = .11010111$, which is known with limited (truncated) precision, maps to $z_1 = .1010111?$. The ? indicates a bit of new information that was previously unknown due to limited knowledge of the initial condition. Subsequent iterations continue to reveal new bits of information, and this apparent capability for a deterministic map to generate new information explains chaos' extreme sensitivity to initial conditions and positive entropy.

To describe reverse-time chaos, we consider the inverse shift map

$$y_{n+1} = \frac{y_n + \sigma_n}{2} \tag{2}$$

for an initial condition $0 \le y_0 < 1$, where $\sigma_n \in \{0,1\}$ is a random sequence. In binary representation, the map (2) defines a right shift followed by inserting σ_n as the new most significant bit. For example, the initial condition $y_0 = .11010111$ maps to y1 = .?1101011, where the ? indicates the new bit of information supplied by σ_1. Thus, the iterated inverse map acts opposite to chaos by absorbing information from a random source. Since precise knowledge of the current state defines all prior iterates, the output of the map is deterministic when viewed in reverse time, and the backward iterates satisfy the chaotic shift map (1). Thus the inverse map (2), which is effectively a linear filter driven by a random process, generates reverse-time chaos.

3. CONTINUOUS REVERSE-TIME CHAOS

To demonstrate reverse-time chaos in a continuous-time linear filter, we consider the driven second-order linear system

$$\ddot{x} + 2\beta\dot{x} + \left(\omega^2 + \beta^2\right)x = s(t) \tag{3}$$

where $x(t)$ is the scalar state, $\beta = \ln 2$ is the decay rate, and $\omega = 2\pi$ is the frequency of the damped oscillations. The input signal $s(t)$ is

$$s(t) = A \cdot s_n, \quad n \le t < n+1 \tag{4}$$

where each $s_n \in \{-1,+1\}$ is random and A is a fixed amplitude. Since equation (1) is linear, we set $A = 1$ without loss of generality. The homogeneous solution of equation (1) is

$$x_h(t) = C \cdot 2^{-t} \cos(2\pi t + \phi) \tag{5}$$

where C and ϕ are integration constants. It is easy to show that the homogenous solution satisfies

$$x_h(t+1) = \frac{1}{2} x_h(t) \tag{6}$$

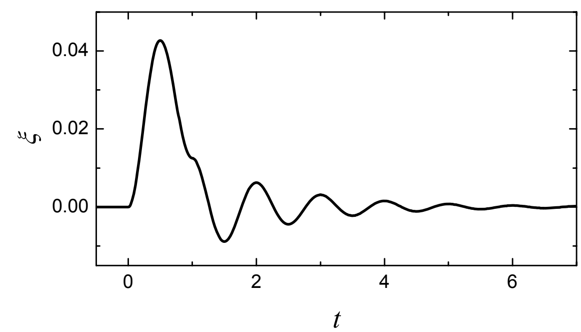

Fig. 1. Unit pulse response.

for all $t \ge 0$; in fact, requiring a solution that satisfies (6) defines the particular values we use here for the system parameters β and ω [Corron *et al.*, 2006].

To solve for a particular solution, we consider the response of the linear system to excitation by a unit pulse. That is, we solve the initial value problem

$$\ddot{\xi} + 2\beta\dot{\xi} + \left(\omega^2 + \beta^2\right)\xi = \begin{cases} 1, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases} \tag{7}$$

subject to the homogenous initial conditions $\xi(0) = \dot{\xi}(0) = 0$. We find

$$\xi(t) = \frac{1}{4\pi^2 + (\ln 2)^2} \begin{cases} 1 - 2^{-t}\left[\cos(2\pi t) + \frac{\ln 2}{2\pi}\sin(2\pi t)\right], & 0 \le t < 1 \\ 2^{-t}\left[\cos(2\pi t) + \frac{\ln 2}{2\pi}\sin(2\pi t)\right], & t \ge 1. \end{cases} \tag{8}$$

We extend this solution to negative time by defining $\xi(t) = 0$ for $t < 0$, and the complete pulse response is plotted in Fig. 1. A general solution to equation (3) is then found by superposition

$$x(t) = \sum_{n=-\infty}^{\infty} s_n \; \xi(t-n). \tag{9}$$

Using the unit pulse response (8), we find

$$x(t) = \frac{s_{[t]} + 2^{[t]-t}\left[\cos(2\pi t) + \frac{\ln 2}{2\pi}\sin(2\pi t)\right] \cdot \left\{-s_{[t]} + \sum_{i=1}^{\infty} 2^{-i} s_{[t]-i}\right\}}{4\pi^2 + (\ln 2)^2} \tag{10}$$

where the notation $[t]$ indicates the largest integer less than or equal to t.

We claim the waveform (10) is chaotic when viewed in reverse time. This claim can be justified using the results of Hirata and Judd, 2005, who derived necessary conditions for the basis pulse function to assure the superposed dynamics are conjugate to a chaotic shift map. Instead, here we directly show that a shift dynamics exists in the general solution (10). To this end, we define a Poincaré return at integer times $t = [t]$. The nth return is then

$$x(n) = \frac{1}{4\pi^2 + (\ln 2)^2} \sum_{i=1}^{\infty} 2^{-i} s_{n-i} \; . \tag{11}$$

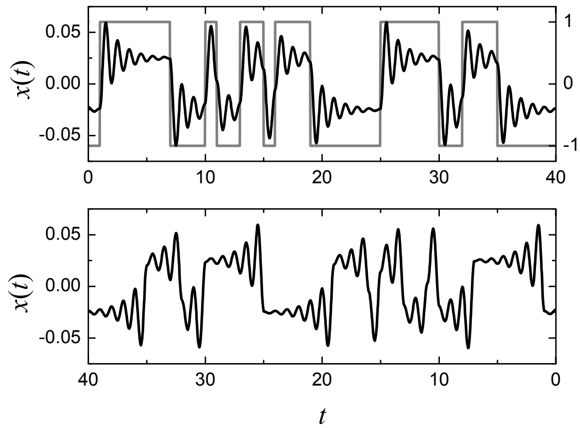

Fig. 2. Forward (top) and reverse (bottom) time waveforms generated by integrating equation (7) for the random sequence $s(t)$ shown in gray.

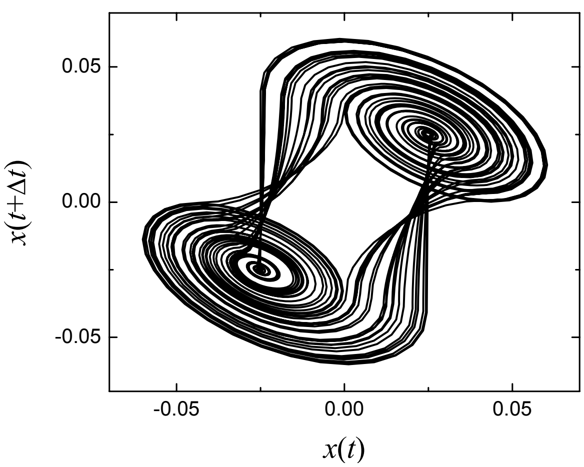

Fig. 3. "Attractor" generated by integrating equation (3) using time delay embedding with $\Delta t = 1/3$.

Defining the scaled return

$$y_n = \frac{4\pi^2 + (\ln 2)^2}{2} x(n) + \frac{1}{2} \tag{12}$$

yields

$$y_n = \sum_{i=1}^{\infty} 2^{-i} \sigma_{n-i} \tag{13}$$

where $\sigma_n = \frac{s_n + 1}{2}$ maps the bipolar symbols $s_n \in \{-1, +1\}$ to the bits $\sigma_n \in \{0, 1\}$. It is easy to verify that successive scaled returns (13) satisfy

$$y_n = 2 y_{n+1} \bmod 1 \tag{14}$$

which is a chaotic shift map backward in time that is equivalent to equation (2). Consequently, the continuous-time waveform (10) exhibits chaos in reverse time. We note this does not imply that integrating equation (3) backward in time will generate a chaotic waveform. Instead, this result implies that the general solution will, when viewed in reverse time, exhibit the properties of a "chaotic" waveform, namely determinism (the shift map) and a positive Lyapunov exponent ($\beta = \ln 2$).

In Fig. 2, we show a solution obtained by integrating equation (3) for a random sequence $s(t)$. The top plot shows a portion of the driving sequence and waveform in forward time. The bottom plot shows the same waveform in reverse time. The reverse-time waveform is similar to chaotic waveforms produced by the Lorenz system [Lorenz, 1963]. In Fig. 3 we show an "attractor" constructed using delay embedding of the waveform with $\Delta t = 1/3$. We note a similarity of the "attractor" in Fig. 3 with certain projections of the butterfly attractor of the Lorenz system.

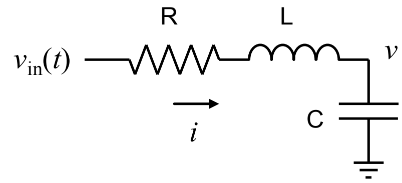

Fig. 4. Driven resistor-inductor-capacitor (RLC) filter circuit for demonstrating reverse-time chaos.

4. ELECTRONIC FILTER CIRCUIT

Significantly, equation (3) models a number of elementary physical systems including, for example, a damped linear pendulum or spring-mass system.

Here we demonstrate reverse-time chaos using the electronic filter shown in Fig. 4. Among the first circuits known to students, it is modeled as

$$LC \frac{d^2 v}{dt^2} + RC \frac{dv}{dt} + v = v_{\text{in}}(t) \tag{15}$$

where $v(t)$ is the voltage across the capacitor. The applied drive voltage $v_{\text{in}}(t)$ is

$$v_{\text{in}}(t) = V_{\text{in}} \cdot s_n, \quad n \le \frac{t}{T} < n+1 \tag{16}$$

where V_{in} is a fixed amplitude, $s_n \in \{-1, +1\}$ is a random sequence, and T is the fundamental drive period. Introducing a dimensionless time $\tau = t/T$ yields

$$\frac{d^2 v}{d\tau^2} + \frac{TR}{L} \frac{dv}{d\tau} + \frac{T^2}{LC} v = f(\tau) \tag{17}$$

where

$$f(\tau) = \frac{T^2 V_{\text{in}}}{LC} \cdot s_n, \quad n \le \tau < n+1\,. \tag{18}$$

We note that the system (17)-(18) is in the same form as system (3)-(4); thus, the circuit can exhibit reverse-time chaos for parameters satisfying

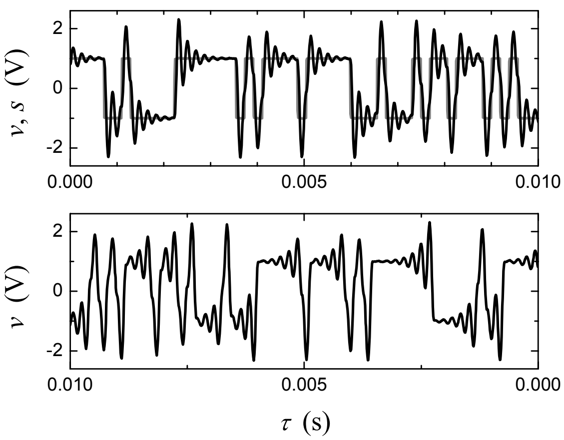

Fig. 5. Forward (top) and reverse (bottom) time waveforms captured from the RLC linear filter circuit for the random bipolar drive signal shown in gray.

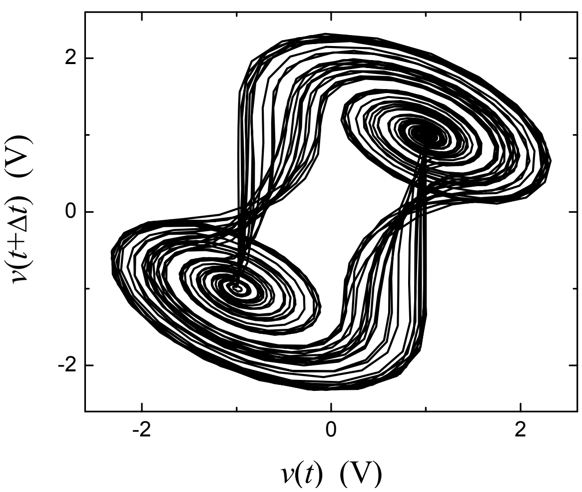

Fig. 6. Measured "attractor" from RLC filter circuit generated by time delay embedding with $\Delta t = 60$ μs.

$$\frac{TR}{L} = 2\ln 2 \tag{19}$$

and

$$\frac{T^2}{LC} = 4\pi^2 + (\ln 2)^2 . \tag{20}$$

We implement the circuit shown in Fig. 4 using discrete electronic components. We use an inductor with $L = 7.5$ mH and intrinsic series resistance $R = 57\ \Omega$. The requirements (19) and (20) then give $T = 180$ μs and $C = 0.11$ μF, respectively. To drive the circuit, we use a digital signal processing card (Innovative Integration ADC64) hosted in a PC to generate random ±1-V bipolar bits at 5.6 kHz. The output waveform $v(t)$ is sampled at 100 kHz using a 12-bit data acquisition card (Keithley DAS-1802).

A typical output waveform captured from the filter is shown in Fig. 5. The top plot shows the input drive signal and output voltage in forward time. The bottom plot shows the same output waveform in reverse time. In Fig. 6 we show the measured "attractor" constructed by delay embedding with $\Delta t = 60$ μs. Again, we note the similarity of the reverse-time waveform and "attractor" to those of the chaotic Lorenz system.

5. DETERMINISM AND RANDOMNESS

The construction of such a simple passive electronic filter that generates reverse-time chaos when driven by a random signal indicates that such phenomena can easily occur in physical systems. Besides allowing possible technological applications, this capability impacts our view of determinism and randomness in nature. From an information point of view, deterministic chaos produces information by amplifying microscopic details of the initial state that were initially beyond the observer's ability to measure [Shaw, 1980]. Even though the unfolding information is new to the observer, the future dynamics are completely determined by the present state. In this way, chaos beautifully reconciles chance and determinism.

Reverse-time chaos turns determinism around: the present state stores the system's entire history. The infinite sequence of prior random pulses can be reconstituted from a perfect knowledge of the present state of the filter. This property—that the present state determines the past—implies determinism when viewed in reverse time. Even though the information used to generate reverse-time chaos is truly random, this information is deterministically mapped to microscopic levels in the system state, eventually beyond the reach of any observer's precision. Seeing further into the past requires ever greater measurement precision, implying a sensitive dependence on initial conditions and a positive Lyapunov exponent in reverse time. In this way, the random reverse-time waveform is entirely consistent with a chaotic, deterministic forward-time waveform.

6. CONCLUSION

In this paper, we have shown that a reverse-time chaotic waveform can be generated by a linear, second-order filter driven by a random source. We believe it is surprising that chaos in any form can be generated by such a remarkably simple system. This alternative method for generating broadband chaotic waveforms may be useful in a number of potential technological applications including spread-spectrum communications, UWB radar and ladar, and the prediction and control of electromagnetic interference in radio frequency circuits. That the language of deterministic chaos provides a meaningful description for signals not generated by a nonlinear dynamical system suggests chaos may be more fundamental than previously supposed. At the very least, the possibility of chaos by such a simple mechanism provides a new perspective on the interplay and reconciliation of deterministic and random processes in nature.

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